Ex. $A = \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Let $x_3 = s \quad x_5 = t$

$x_4 = -4t \quad , \quad x_2 = s + 2t \quad , \quad x_1 = t - 2s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} t-2s \\ s+2t \\ s \\ -4t \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

The spanning set of the kernel = $\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$

How to find the range of a matrix transformation

$\begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ → from previous example

Spanning set of the range is = $\left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}}_{\text{Same columns of the pivot but from original matrix}} \right\}$

Theorem: The pivot columns of a matrix $\underline{A}$ form a basis for the range of the linear transformation

Ex. $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Basis of the range is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\}$

The nullity and the rank of linear transformation

Let $T: V \to W$ be a linear transformation. The dimension of the kernel of T is called the nullity of T. The dimension of the range of T is called the rank of T

Special case: In case of matrix transformation $T(V) = A\underline{V}$

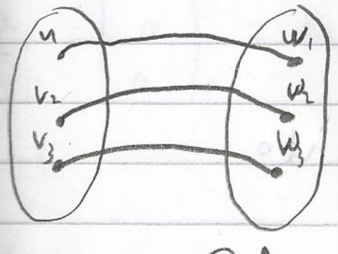
$\text{rank}(T) = \text{rank}(A)$

$\text{nullity}(T) = \text{no. of free variables}$

Ex. $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ → from previous example

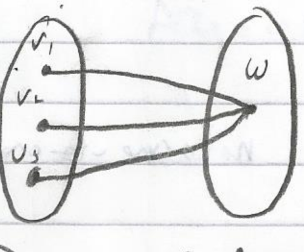
rank = 3, nullity = 2

One-to-one and onto linear transformation



Domain Codomain

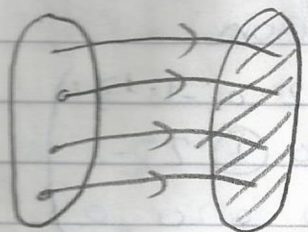
one-to-one



Domain Codomain

not one-to one

if $V_1 \neq V_2 \Longleftrightarrow w_1 \neq w_2$



range = Codomain
(Onto)

Ex. $\begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix} \rightarrow$ echelon form

Consistent $\Rightarrow$ onto (~~not one-to-one~~)
free variable exist $\Rightarrow$ not one-to-one

__Theorem__: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if the equation $T(x) = 0$ has only the trivial solution [no free variables]

__Theorem__: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is onto if and only if the rank of T is equal to m (no. of rows)

Ex. a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ one-to-one onto

b) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ one-to-one not onto

c) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ not one-to-one onto

d) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ not one-to-one not onto